A Vertex at the End of the Rainbow

Some Results on Rainbow Connection Numbers in Graphs

By Daniel Vaughn

Under the direction of
Dr. John Caughman
With second reader
Dr. Bin Jiang

In partial fulfillment of the requirement for the degree of
Masters of Science in Mathematics

Portland State University
Fariborz Maseeh Department of Mathematics and Statistics
December 08, 2014

Table of Contents

Introduction

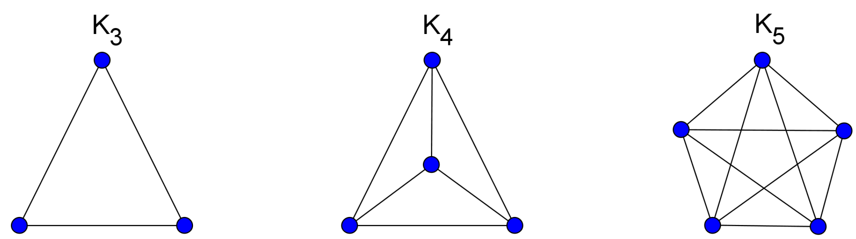
In this paper, we present results found in the 2008 article "Rainbow Connection Numbers in Graphs" by Gary Chartrand, Garry L. Johns, Kathleen A. McKeon and Ping Zhang [1].

Specifically we follow their presentation of results concerning the rainbow connection numbers and strong rainbow connection numbers of several families of graphs. We begin with the necessary definitions.

We define a graph G to be a set of vertices $V(G) = \{v_1, v_2, ..., v_n\}$ together with a set of edges $E(G) = \{e_1, e_2, ..., e_m\}$. A **finite** graph has finite vertex and edge sets. A **simple** graph contains no multiple edges or loops, so that any edge in a simple graph adjoins exactly two distinct vertices $v_i, v_j \in V(G)$. Thus if G is a simple graph and $e \in E(G)$, we can write $e = v_iv_j$ for some $v_i, v_j \in V(G)$. In this case, we say v_i and v_j are **adjacent**. We say that G is **undirected** provided $v_iv_j \in E(G)$ if and only if $v_jv_i \in E(G)$, for any distinct $v_i, v_j \in V(G)$. A **path** between vertices u and v in a graph G (also called a **u-v path**) consists of a sequence of vertices $v_1, v_2, ..., v_n$ with $v_1 = u$, $v_n = v$ and $v_iv_{i+1} \in E(G)$ for $1 \leq i \leq n-1$. Finally, a graph G is **connected** if, for any vertices $u, v \in V(G)$, there exists a $u - v$ path in G. In this paper, we work only with finite, simple, undirected, connected graphs. We look now at some common families of graphs which we will refer to frequently.

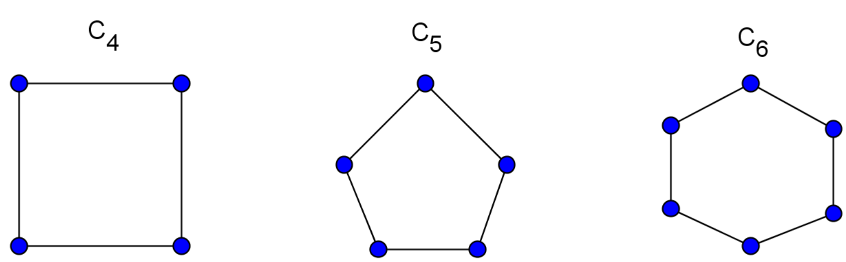
1.1 The Complete Graphs

In a **complete graph**, each pair of vertices in $V(G)$ are adjacent. The complete graphs are denoted by K_n where $n = |V(G)|$.



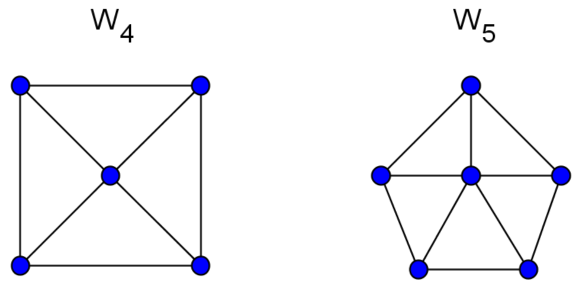
1.2 The Cycles

A **cycle** is a sequence of vertices $v_1, v_2, ..., v_k$ where the vertices are distinct and v_i is adjacent to v_j if and only if $|i - j| \equiv 1 \pmod{k}$. A cycle on k vertices is referred to as C_k.



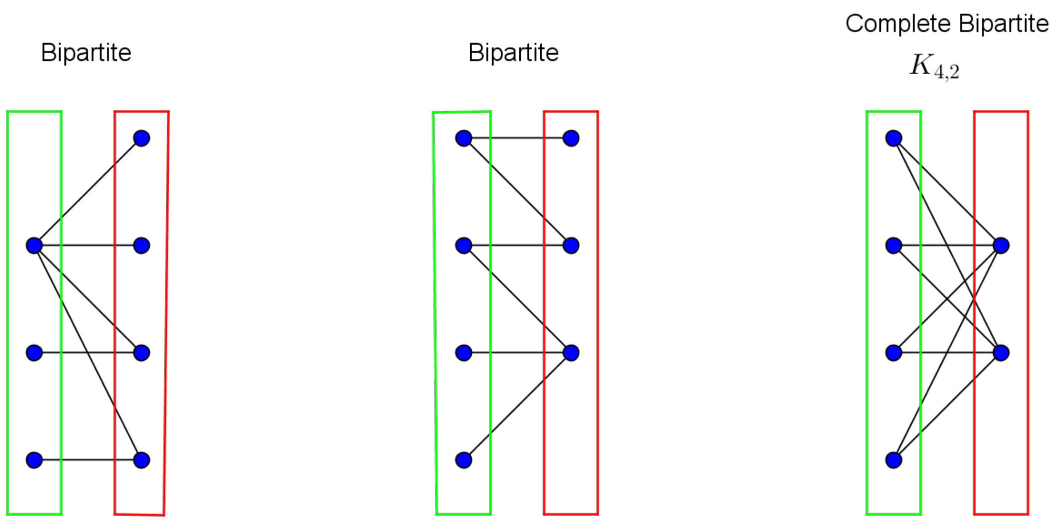
1.3 The Wheels

A **wheel** can be formed as follows: Start with the cycle C_k. Take a new vertex v and create an edge between v and each vertex in C_k. The new graph is the wheel W_k.



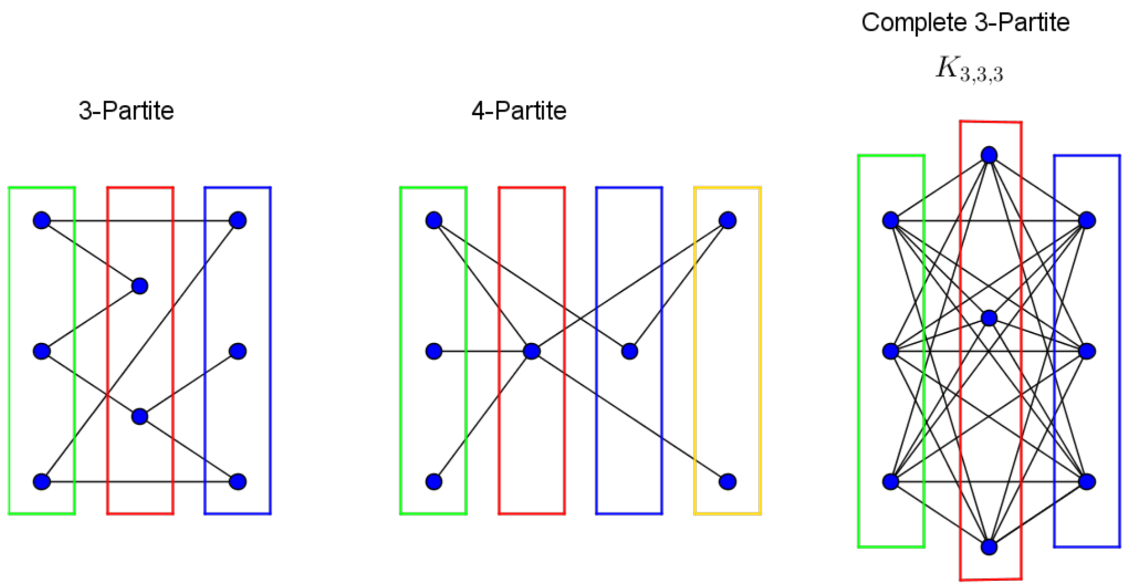
1.4 Bipartite Graphs

A set of pairwise nonadjacent vertices is called an **independent set**. Suppose the vertices of graph G can be partitioned into exactly two independent sets A and B. Then each vertex in A is adjacent only to vertices in B and vice-versa. In this case we call G **bipartite**. Moreover if <u>each</u> vertex in A is adjacent to <u>each</u> vertex in B we say that G is a **complete bipartite** graph, denoted by $K_{s,t}$ where $|A| = s$ and $|B| = t$. It is convenient to display independent sets framed separately, as below.



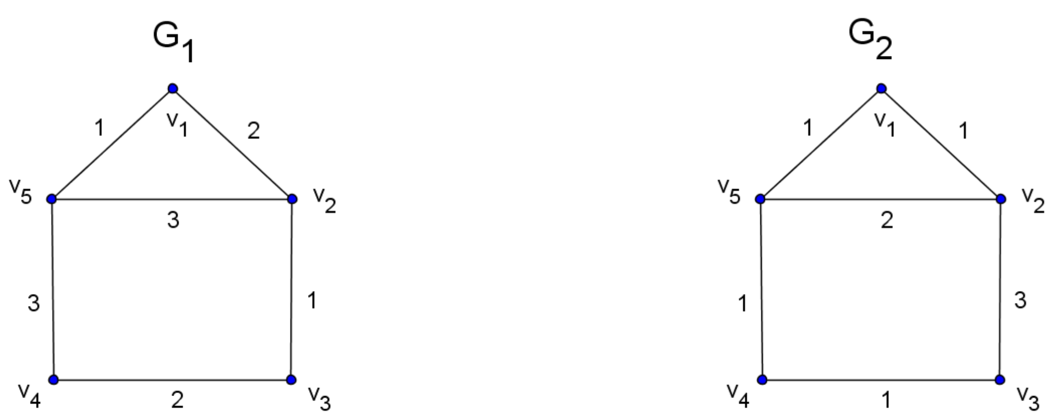
• 1.5 Multipartite Graphs

A graph is called **multipartite** if it can be partitioned into more than one independent set. A **k-partite** graph can, in particular, be partitioned into k independent sets. A **complete k -partite** graph can be partitioned into k independent sets $S_1, S_2, ..., S_k$ such that each vertex in S_i is adjacent to every vertex in G except those in S_i, for $1 \leq i \leq k$. In this case the graph could be denoted by $K_{n_1,n_2,...,n_k}$, where $n_i = |S_i|$ for each i.



2. Rainbow Colorings

Let G be a graph. A **coloring** $c : E \to \{1, 2, ..., k\}$, $k \in \mathbb{N}$, is a function which assigns to each edge in G exactly one color i such that $1 \leq i \leq k$. Incident edges (edges which share an endpoint) are allowed to carry the same color in the general case. The so-called "house" graphs below demonstrate two possible colorings on the same graph.



Given a coloring of a graph G, a path P is a **rainbow path** if no two edges of P are colored the same. So in the coloring of graph G_1, for example, $v_2v_3v_4$ is a rainbow path but $v_2v_5v_4$ is not. A colored graph G is **rainbow connected** if G contains a rainbow $u-v$ path for every two vertices u and v of G. In this case, the coloring c is called a **rainbow coloring** of G. It can be shown easily by exhaustion that any pair of vertices in G_1 has a rainbow path between them, which means G_1 is rainbow connected. On the other hand, G_2 is not rainbow connected since any path between v_1 and v_4 contains two edges colored 1.

The minimum number of colors on a graph G which provides a rainbow coloring is called the **rainbow connection number rc(G)** of G. A rainbow coloring on G using exactly $rc(G)$ colors is a **minimum rainbow coloring** of G.

A **geodesic** in G is any shortest path between two vertices u and v. Given a coloring of a graph G, a **rainbow u-v geodesic** in G is a rainbow $u-v$ path of length $d(u,v)$ where $d(u,v)$ is the distance (path length) between u and v in G. A colored graph G is **strongly rainbow connected** if a rainbow $u-v$ geodesic exists for any pair of vertices u, v in G, in which case the coloring is a **strong rainbow coloring**. The minimum number of colors required for a strong rainbow coloring is the **strong rainbow connection number src(G)**.

2.1 Elementary Bounds

It should be clear that any graph which is strongly rainbow connected is also rainbow connected, since any rainbow geodesic is a rainbow path. Thus any strong rainbow coloring is also a rainbow coloring, though the converse does not always hold. Consider too that $rc(G)$ and $src(G)$ are both defined for any connected graph G, for if $m = |V(G)|$ we could simply use m colors on G, one for each edge. Thus any paths in G, including geodesics, would be rainbow paths. We can then say that $rc(G) \leq src(G)$ for any connected graph G.

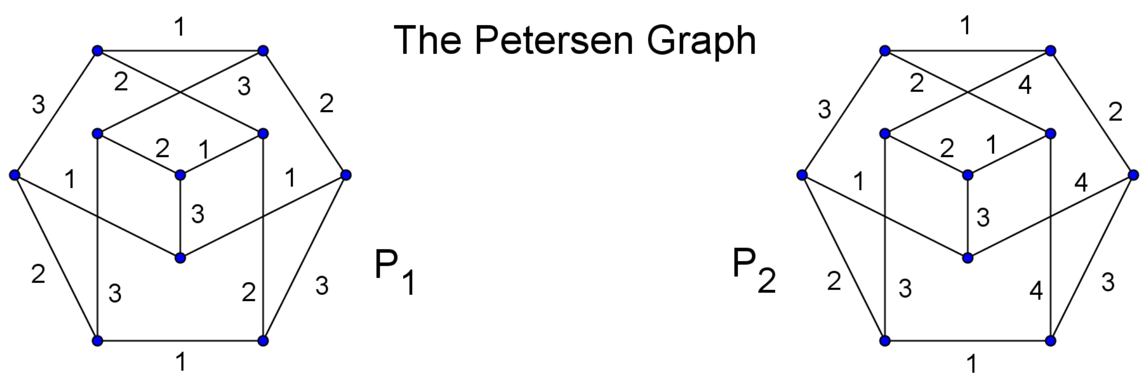
The length of the longest geodesic in G (the longest shortest path) is the **diameter** of G, denoted $diam(G)$. Since $diam(G)$ is the length of a geodesic between some pair of vertices in $V(G)$, $diam(G)$ serves as a lower bound on both rainbow connection numbers. Since each edge may only be assigned 1 color, a natural upper bound on the rainbow connection numbers $rc(G)$ and $src(G)$ is $|E(G)| = m$. Then the following holds:

$$(1) \qquad diam(G) \leq rc(G) \leq src(G) \leq m$$

We will reference the above inequality frequently throughout the paper.

2.2 The Petersen Graph

By way of example we find the rainbow connection numbers of the Petersen graph, which we will call P. Observe that P_1 demonstrates a rainbow 3-coloring on P, which means $rc(P) \leq 3$. But $rc(P) \geq 2$, since there exist $u, v \in V(P)$ with $d(u,v) \neq 1$. Suppose $rc(P) = 2$ to gain a contradiction. It is known that the length of the shortest cycle in P is 5, which is odd. So there must be two incident edges with the same color assigned. Call these $e_1 = uv$ and $e_2 = vw$. The path $T = u, v, w$ is the only $u - w$ path of length 2 (since a second distinct $u - w$ path length 2 would create a 4 cycle in P, which contradicts the fact that P has no cycles of length less than 5). But e_1 and e_2 share a color, meaning T is not a rainbow path. Any other $u - v$ path has length 3 or greater and thus is not a rainbow path, since we assumed $rc(P) = 2$. We conclude a 2-coloring is impossible, meaning $rc(P) = 3$.



We next consider $src(P)$. From (1), we know $src(P) \geq rc(P) = 3$. Graph P_2 from the previous page demonstrates a strong rainbow 4-coloring of P, so $src(P) \leq 4$. The edge chromatic number of the Petersen graph is commonly known to be 4, indicating that there is no 3-coloring of the edges of P where incident edges always carry distinct colors. As we argued above, though, having incident edges that share a color in the Petersen graph forces existence of two vertices with no rainbow geodesic between them. So $src(P) \neq 3$, leaving $src(P) = 4$ as the only possibility.

2.3 Extreme Cases

Proposition 1.1: Let G be a nontrivial connected graph of size m. Then:

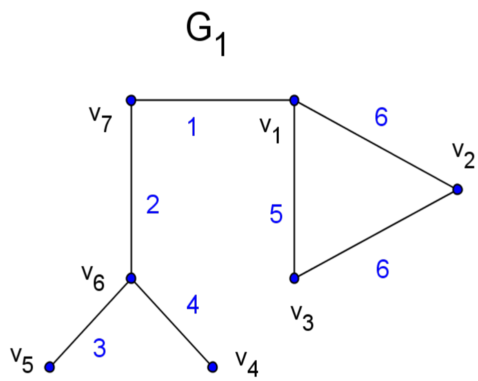
(a) $src(G) = 1$ if and only if G is a complete graph,
(b) $rc(G) = 2$ if and only if $src(G) = 2$,
(c) $rc(G) = m$ if and only if G is a tree.

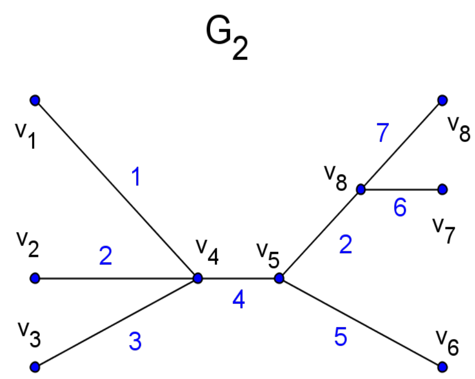
Proof of (a): If G is a complete graph, then $d(u, v) = 1$ for any $u, v \in V(G)$. Since an edge is always a rainbow geodesic in G, we may use the coloring $c(e_i) = 1$ for each $e_i \in E(G)$, yielding that $src(G) = 1$. Note that if $src(G) = 1$, each geodesic in G must have length 1. So G is a complete graph.

Proof of (b): Now suppose $rc(G) = 2$. Then $src(G) \geq 2$ by (1). Since $rc(G) = 2$, then for any nonadjacent $u, v \in V(G)$, there must exist a rainbow $u - v$ path length 2. Such a path must be a geodesic, since u, v are not adjacent, so it is a rainbow geodesic.

Notice if u and v are adjacent, we again have a rainbow $u-v$ geodesic. Thus $src(G)=2$. For the converse, assume $src(G)=2$ and observe, by (1), that $rc(G)\leq 2$. Consider that $rc(G)=1$ would require that each rainbow path in G have length at most one. This only occurs in complete graphs. However, if G were a complete graph we would have $src(G)=1$, which contradicts our operating assumption. We conclude, then, that $rc(G)=2$.

Proof of (c): We use the contrapositive and suppose G is not a tree. This means there is a cycle $C=v_1,v_2,...v_k,v_1$. Use $(m-2)$ distinct colors for all edges of G except v_1v_2 and v_2v_3, for which we use the same color. Refer to graph G_1 for a small example.





This defines an $(m-1)$-coloring on G. Any v_i-v_2 path in G need only include one among edges v_1v_2 and v_2v_3, meaning it is a rainbow path. All other v_i-v_j paths need not include v_2, since it is part of a cycle. So $rc(G)\neq m$. Next, supposing G is a tree with m edges, consider the fact that each v_iv_j path in G is unique. See the graph of G_2 for another small example. If G could be rainbow colored with fewer than m colors, there would, by the Pigeonhole Principle, be two edges e_s,e_t which share a color, say 2, without loss of generality. Since each path in G is distinct, there is a v_i-v_j path including e_s,e_t. This contradicts that we had a rainbow coloring on fewer than m colors. Thus $rc(G)=m$. $\square$

3. Cycles and Wheels

We continue with a result regarding cycles. Recall that a cycle on n vertices, C_n, denotes a sequence of vertices $v_1,v_2,....,v_n,v_1$ where the v_i are distinct if $i\neq 1$. In general, for cycles on n vertices we will use the labeling $e_i=v_iv_{i+1}$ for $1\leq i\leq(n-1)$ and $e_n=v_nv_1$.

3.1 Connection Numbers on Cycles

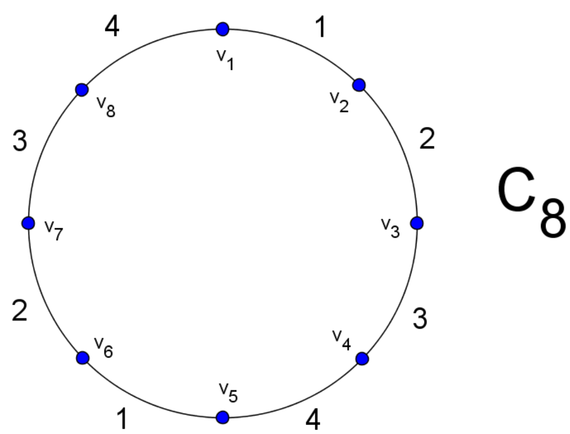
Proposition 2.1:

For each integer $n\geq 4$, $rc(C_n)=src(C_n)=\lceil\frac{n}{2}\rceil$.

Proof: Label the vertices of C_n by $v_1, v_2, ...v_n$.

Case n is even: Then $n = 2k$ for some $k \in \mathbb{Z}$. Define the coloring c_0 by $c_0(e_i) = i$ for $1 \leq i \leq k$ and $c_0(e_i) = i - k$ if $k+1 \leq i \leq n$. This coloring ensures that $c_0(e_i) = c_0(e_j)$ only if $i - k \equiv j \pmod{n}$. Thus edges with the same color have $(k-1)$ edges between them. So any path length k or less will never repeat edges. Because $diam(C_n) = k$, each geodesic in C_n then defines a rainbow path.

So c_0 is a strong rainbow coloring, meaning $rc(C_n) \leq src(C_n) \leq k = \frac{n}{2} = \lceil \frac{n}{2} \rceil$. However, using (1), we know $\frac{n}{2} = diam(C_n) \leq rc(C_n) \leq src(C_n)$. Therefore $rc(C_n) = src(C_n) = \lceil \frac{n}{2} \rceil$



Example for n = 8, k = 4

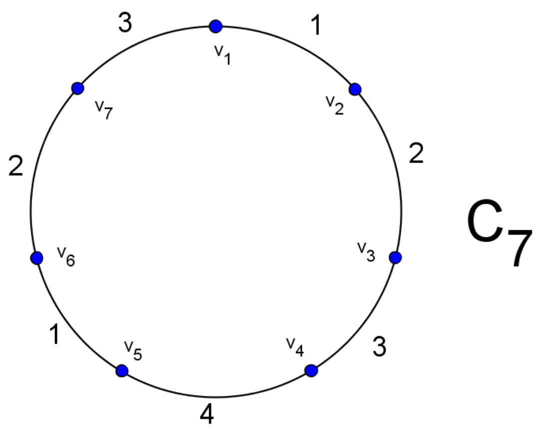
Case n is odd: Then $n = 2k+1$ for some $k \in \mathbb{Z}$, $k \geq 2$. Define the coloring $c_1(e_i) = i$ for $1 \leq i \leq k+1$ and $c_1(e_i) = i - k - 1$ if $k+2 \leq i \leq n$.

From (1) we see that $src(C_n) \geq rc(C_n) \geq diam(C_n) = k$. We show first that c_1 defines a strong rainbow $(k+1)$ coloring on C_n, by which it is also a $(k+1)$ rainbow coloring. We finish by proving that $rc(C_n) \neq k$, forcing $rc(C_n) = src(C_n) = k+1$.

In a similar manner to the case where n is even, consider the case where $c_1(e_i) = c_1(e_j)$. This occurs either when $i - (k+1) \equiv j \pmod{n}$ or $i + k \equiv j \pmod{n}$. So any edges in $E(C_n)$ which share a color have at least $(k-1)$ edges between them. Pick any vertices $u, v \in V(C_n)$. Since $d(u,v) \leq k$ we are guaranteed a rainbow $u - v$ geodesic. Thus c_1 is a strong rainbow coloring.

Now, assume by way of contradiction that $rc(C_n) = k$, calling this coloring c^*. Consider two vertices u and v with $d(u,v) = k$. Then one $u - v$ path is a geodesic, using all k colors from c^* and the other $u - v$ path has length $k+1$ and is not a rainbow path. Without loss let $c^*(v_{k+1}v_{k+2}) = k$. Since paths $P_1 = v_1, v_2, ..., v_{k+1}$ and $Q_1 = v_1, v_n, v_{n-1}, ..., v_{k+2}$ are both rainbow geodesics, they each cross an edge with color k. Notice that the paths $P_2 = v_2, v_3, ..., v_{k+2}$ and $Q_2 = v_n, v_{n-1}, ..., v_{k+1}$ are both rainbow geodesics. Because $c^*(v_{k+1}v_{k+2}) = k$, we are forced to conclude that $c^*(v_1v_2) = c^*(v_1v_n) = k$. We arrive at a contradiction, since there is now no

$v_2 - v_n$ path in C_n. Therefore $rc(C_n) \neq k$. Then $rc(C_n) = src(C_n) = k + 1 = \lceil \frac{n}{2} \rceil$. $\square$



Example for n = 7, k = 4

3.2 Rainbow Connection Numbers of the Wheels

We earlier defined a family of graphs called the wheels, denoted W_n. To construct the wheel W_n, begin with the cycle C_n and join each vertex in C_n to a new vertex v_0. We follow with some results on rainbow connection numbers of wheels.

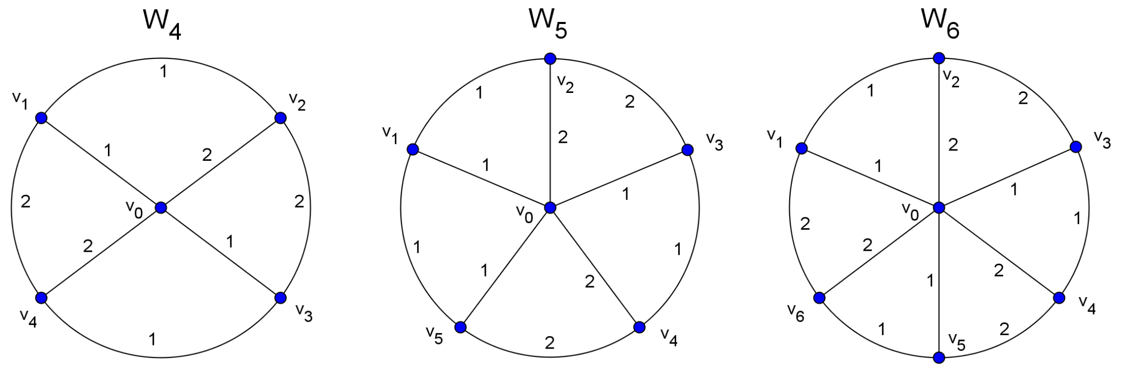
Proposition 2.2: For $n \geq 3$, the rainbow connection number of W_n is:

$$rc(W_n) = \begin{cases} 1 & \textit{if } \ n = 3 \\ 2 & \textit{if } \ 4 \leq n \leq 6 \\ 3 & \textit{if } \ n \geq 7 \end{cases}$$

Proof: Since $W_3 = K_4$, it is immediate that $rc(W_3) = 1$ (Prop 1.1). Now suppose $4 \leq n \leq 6$. Then $rc(W_n) \geq 2$ since W_n is not complete for $n > 3$. Let $c : E(W_n) \to \{1, 2\}$ be defined as follows:

- $c(v_i v_0) = 1$ if i is odd and $c(v_i v_0) = 2$ if i is even.
- $c(v_i v_{i+1}) = 1$ if i is odd and $c(v_i v_{i+1}) = 2$ if i is even

The colorings for $4 \leq n \leq 6$ are shown next. As always, there is a rainbow path (an edge) from the center vertex v to any vertex of the subgraph C_n, so we need only check that there is a rainbow $u - v$ path of length 2 between any pair of vertices in C_n to verify c is indeed a rainbow 2-coloring.



Take $u, v \in W_4$. Notice $d(u,v) \leq 2$ and no incident edges in C_4 share a color. Hence there is a rainbow $u-v$ path, meaning c is a rainbow 2-coloring on W_4. If u, v are vertices in W_5 we again have a rainbow $u-v$ path in C_5 except in the case of vertices v_2, v_5. But v_5, v_0, v_2 is a rainbow $v_5 - v_2$ path, so c is a rainbow 2-coloring on W_5. Finally suppose $u, v \in W_6$. If $d(u,v) \leq 2$ in C_6 there is a rainbow $u-v$ path, similar to W_4. If $d(u,v) = 3$, then $c(v_0u) \neq c(v_0v)$, since $u \not\equiv v \pmod 2$. Thus u, v_0, v is a rainbow $u-v$ path as well, and c is a rainbow 2-coloring on W_6.

If $n \geq 7$, consider the coloring $c : E(W_n) \to \{1, 2, 3\}$ defined by:

- $c(v_iv_0) = 1$ if i is odd,
- $c(v_iv_0) = 2$ if i is even,
- $c(e) = 3$ for each $e \in E(C_n)$

Take $v_i, v_j \in C_n$, where without loss $j > i$. If u, v are adjacent, then the edge v_iv_j is a rainbow path. Otherwise $d(v_i, v_j) = 2$. Since the spokes of the wheel alternate color, we have a rainbow 2-path from v_j to either v_i or v_{i+1}. If the first case, we are done. Otherwise v_j, v_0, v_{i+1}, v_i is a rainbow $v_j - v_i$ path of length 3. Thus $rc(W_n) \leq 3$ if $n \geq 7$. Since W_n is not the complete graph for $n > 3$, we have that $rc(W_n) \geq 2$.

Assume by way of contradiction that $rc(W_n) = 2$, so that a rainbow 2-coloring c_0 exists. Without loss of generality, say that $c_0(v_1v_0) = 1$. Then for $4 \leq i \leq (n-2)$, the only $v_1 - v_i$ path of length 2 is v_1, v_0, v_i. This means $c_0(v_0v_i) = 2$ for $4 \leq i \leq (n-2)$. There is only one $v_4 - v_n$ path of length 2, so $c_0(v_nv_0) = 1$. Similarly, there is only one $v_3 - v_n$ path of length 2, which means $c_0(v_3v_0) = 2$, forcing $c_0(v_{n-1}v_0) = 1$. This in turn forces $c_0(v_2v_0) = 2$. There is now no rainbow $v_2 - v_5$ path of length 2 in W_n, contradicting our assumption of a rainbow 2-coloring. We conclude that $rc(W_n) = 3$ if $n \geq 7$, which completes the proof of Proposition 2.2. □

3.3 Strong Rainbow Connection Numbers of the Wheel

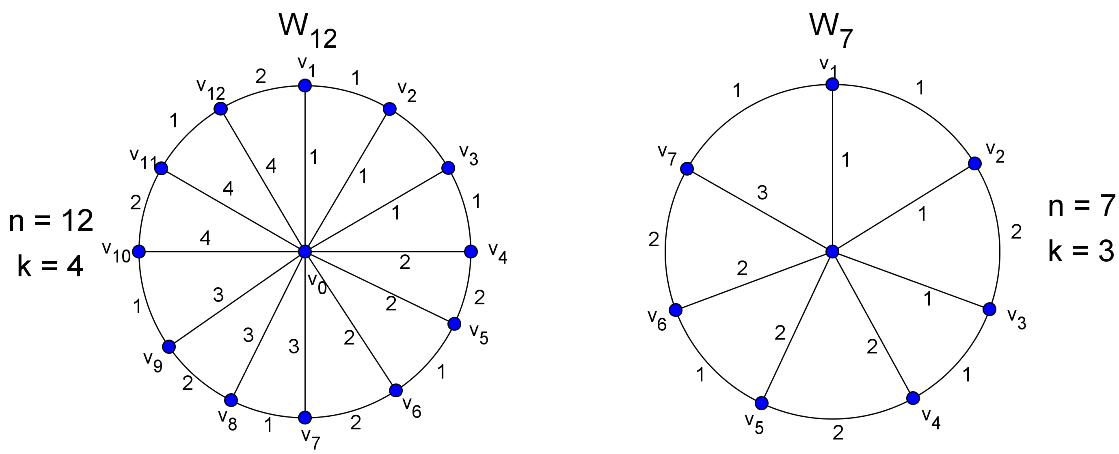
We continue with a result on the strong rainbow connection number of all wheels.

Proposition 2.3: For $n \geq 3$, the strong rainbow connection number of of W_n is:

$$src(W_n) = \lceil \tfrac{n}{3} \rceil$$

From Proposition 1.1 we see that $src(W_3) = 1$. From Proposition 2.2, we have that $rc(W_n) = 2$ for $4 \leq n \leq 6$. Proposition 1.1 then yields that $src(W_n) = 2$ for $4 \leq n \leq 6$. So we safely assume that $n \geq 7$. Then $3k - 2 \leq n \leq 3k$ for some $k \in \mathbb{N}$. Note that it will be sufficient to show that $src(W_n) = k$. We show first that $src(W_n) \geq k$. Assume to the contrary that there exists a $(k-1)$-coloring c on W_n. The **degree** of a vertex v, denoted $deg(v)$, in a graph G is the number of vertices to which v is adjacent in G. Because $deg(v_0) = n > 3(k-1)$, there must exist some set $S \subset V(C_n)$ with $|S| = 4$ and all edges in $\{v_0u : u \in S\}$ colored the same (pigeonhole principle). Since $n \geq 7$, there must exist $u_1, u_2 \in S$ with $d(u_1, u_2) \geq 3$ in C_n and $d(u_1, u_2) = 2$ in W_n. So the only $u_1 - u_2$ geodesic in W_n is u_1, v_0, u_2 This is not a rainbow geodesic since $c(u_1v_0) = c(u_2v_0)$. Hence a $(k-1)$-coloring is not possible, and $src(W_n) \geq k$.

Examples of strong rainbow k-colorings for W_{12} and W_7



Now observe the strong rainbow k-coloring $c : E(W_n) \to \{1, 2, ..., k\}$ defined by:

$$c(e) = \begin{cases} 1 & \textit{if} \ \ e = v_iv_{i+1} \text{ and i is odd,} \\ 2 & \textit{if} \ \ e = v_iv_{i+1} \text{ and i is even,} \\ j+1 & \textit{if} \ \ e = v_iv_0 \ \text{ for } i \in \{3j+1, 3j+2, 3j+3\} \text{ and } 0 \leq j \leq k-1 \end{cases}$$

In the C_n subgraph, edges alternate with the one exception that $c(v_nv_1) = c(v_1v_2)$ if n is odd. The spokes of W_n are colored in groups of three, beginning with v_1, and using colors 1 to $(k-1)$. Take $u, v \in C_n$. If u and v are adjacent, the edge between them is a rainbow geodesic. If $d(u, v) = 2$ in C_n and n is even, then there is a rainbow $u - v$ geodesic in C_n. If n is odd, there is a rainbow 2-path in C_n provided u and v are not the vertices v_n and v_2. In this special case, notice that $c(v_nv_0) \neq c(v_2v_0)$ and hence that $\{uv_0v\}$ is a rainbow geodesic in W_n. Now suppose $d(u, v) \geq 3$ in C_n. Then $c(uv_0) \neq c(vv_0)$, which means $\{u, v_0, v\}$ is a rainbow $u - v$ geodesic in W_n. So c is a rainbow k-coloring, and $src(W_n) \leq k$ as desired. We showed $src(W_n) \geq k$ above, which means $src(W_n) = k = \lceil \frac{n}{3} \rceil$. $\square$

4. Complete Multipartite Graphs

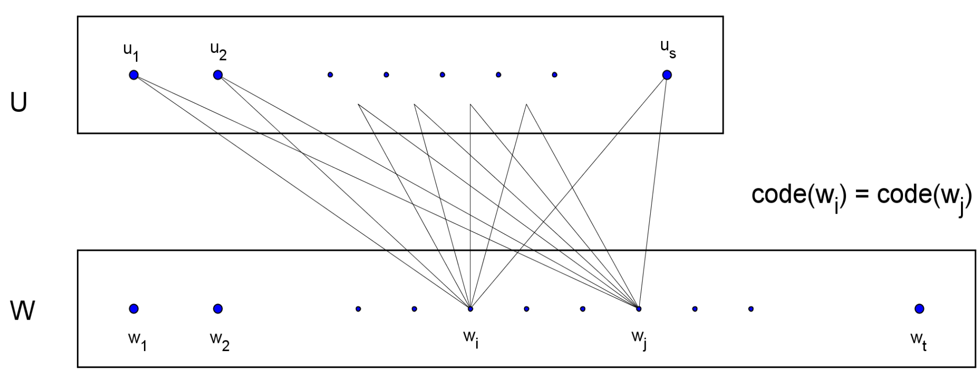
We move now to complete multipartite graphs, which we defined in Section 1.3. We begin with the strong rainbow connection numbers of the complete bipartite graphs. Recall that a complete bipartite graph can be partitioned into two independent sets S and T such that each vertex $s \in S$ is adjacent to every vertex $t \in T$.

4.1 Strong Rainbow Connection in Complete Multipartite Graphs

Theorem 2.4: For integers s and t with $1 \leq s \leq t$,

$$src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$$

Proof: For $s = 1$, $K_{1,t}$ is a tree with t edges (also the familiar star graph on $t+1$ vertices). From Proposition 2.1, then, we have that $src(K_{1,t}) = t = \lceil \sqrt[1]{t} \rceil$. Thus Theorem 2.4 holds for $s = 1$, so assume that $s \geq 2$. Let $\lceil \sqrt[s]{t} \rceil = k$. Then we have: $k - 1 < \sqrt[s]{t} \leq k$.
$\Rightarrow$ $(k-1)^s < t \leq k^s$
$\Rightarrow$ $(k-1)^s + 1 \leq t \leq k^s$ (2)
As before, we first show $src(K_{s,t}) \geq k$ and then display a strong rainbow k-coloring of $K_{s,t}$ to show $src(K_{s,t}) \leq k$. Assume by way of contradiction that $src(K_{s,t}) < k$. Then there is a strong rainbow $(k-1)$-coloring of $K_{s,t}$. Call this coloring c and let U and W be the bipartitions of size s and t respectively. Each vertex in W is adjacent to each of the s vertices in U. For $w \in W$, the s colors assigned to edges connected to w form a natural s-tuple which we will call a color code for w. For example, notationally we write $code(w_i) = (a_1, a_2, ..., a_s)$, where $a_i \in \{1, 2, ...(k-1)\}$ for $1 \leq i \leq s$. By the multiplication rule of counting, there are at most $(k-1)^s$ color codes possible. Using the pigeonhole principle and inequality (2) above, there exist distinct $w_i, w_j \in W$ such that $code(w_i) = code(w_j)$.

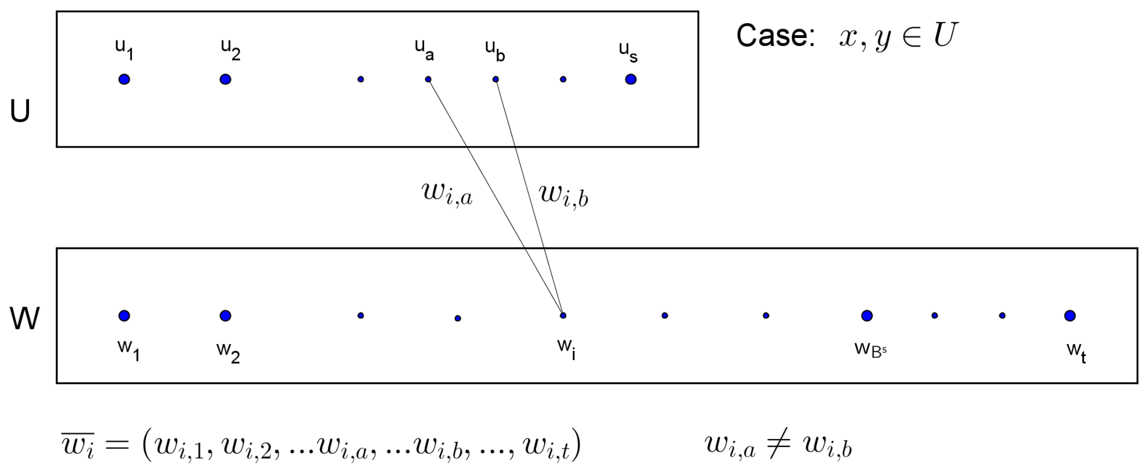


Now, a $w_i - w_j$ geodesic in $K_{s,t}$ is of the form w_i, u, w_j for some $u \in U$. But since w_i, w_j have identical codes, $c(w_i u) = c(w_j u)$ for every $u \in U$. Thus there is no rainbow $w_i - w_j$ geodesic. So our assumption of a $(k-1)$ coloring is false, meaning that $src(K_{s,t}) \geq k$. To show that $src(K_{s,t}) \leq k$, we construct a k-coloring on $K_{s,t}$. Let $A = \{1, 2, ..., k\}$ and $B = \{1, 2, ..., (k-1)\}$, and let A^s and B^s denote the respective cartesian products of sets A and B. We have $|B^s| < t \leq |A^s|$. We label the first $|B^s|$ elements of W as follows:

For $1 \leq i \leq |B^s|$, let $\overline{w_i} = (w_{i,1}, w_{i,2}, ..., w_{i,s})$ be any enumeration of all s-tuples in B^s, where each $w_{i,j}$ represents a color, so that $1 \leq w_{i,j} \leq k$. These s-tuples are the color codes assigned to the first $|B^s|$ elements of W. From the remaining elements of A^s we arbitrarily choose distinct s-tuples as labels w_i, with $|B^s| \leq i \leq t$, for the unlabeled vertices of W. Now use these labels to define the coloring $c : E(K_{s,t}) \to \{1, 2, ..., k\}$ by:

$$c(w_i, u_j) = w_{i,j} \text{ where } 1 \leq i \leq t \text{ and } 1 \leq j \leq s$$

Note that the labels now serve as color codes, where each color code is a distinct
s-tuple from A^s. Now we verify that c is indeed a strong rainbow k-coloring. Pick $x, y \in K_{s,t}$. If $x \in U$ and $y \in W$ (or vice-versa), then the edge xy is a rainbow $x - y$ geodesic. Next let $x, y \in W$ with $x = w_a$, $y = w_b$. Since each of the color codes for vertices in W are distinct, then there exists $u \in U$ such that the u-coordinate of $code(w_a)$ is different from the u-entry of $code(w_b)$. For such a vertex u, we have the rainbow $w_a - w_b$ geodesic w_a, u, w_b.



Finally, assume that $x, y \in U$, with $x = u_a$, $y = u_b$ and (without loss) $a < b$. Recall that in our labeling of the first $|B^s|$ elements of W we used all possible s-tuples from B^s. This guarantees that there exists some $w_i \in W$ such that $w_{i,a} \neq w_{i,b}$. So u_a, w_i, u_b is a rainbow geodesic in $K_{s,t}$. Therefore c is a strong rainbow k-coloring.
$\Rightarrow \quad src(K_{s,t}) = k. \quad \square$

We continue with strong rainbow connection numbers, but shift focus from complete bipartite graphs to the more general complete multipartite graphs. It can be helpful to think of a complete multipartite graph with t independent sets $s_1, s_2, ...s_t$ as a copy of the complete graph K_t where each vertex $v_i \in V(K_t)$ represents the independent set s_i.

Theorem 2.5: Let $G = K_{n_1,n_2,...,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq .... \leq n_k$. Further, let $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then:

$$src(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \lceil \sqrt[s]{t} \rceil & \text{if } s \leq t \end{cases}$$

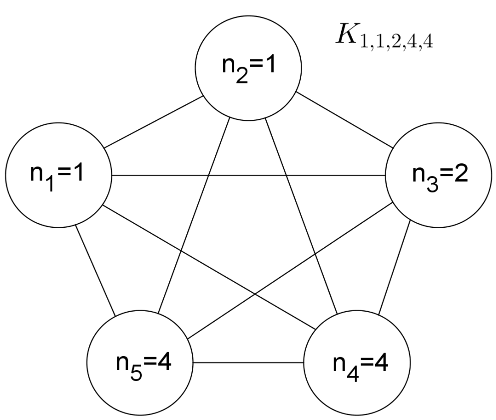
Proof: Let $n = \sum_{i=1}^{k} n_i$. If $n_k = 1$, then $n_i = 1$ is forced for $1 \leq i \leq k$. But this describes the complete graph K_k. From Proposition 1, we have then that $src(G) = 1$. Next consider the case $n_k \geq 2$ and $s > t$. Since there exists a nontrivial independent set, $G \neq K_k$, which means $src(G) \geq 2$. It will be sufficient to show a rainbow 2-coloring to prove that $src(G) \leq 2$. We partition $S = \{n_1, n_2, ..., n_k\}$ into 2 multisets:

$$A = \{a_1, a_2, ..., a_p\} \text{ and } B = \{b_1, b_2, ..., b_q\}$$

with the restrictions that:

1) $p + q = k$
2) $a = \sum_{i=1}^{p} a_i \leq \sum_{j=1}^{q} b_j$
3) $b - a$ is minimized with $b - a \geq 0$

From this point forward, when illustrating complete multipartite graphs, we use the idea of a complete graph where each vertex represents all vertices of an independent set from G. An edge in K_k between n_i and n_j represents an edge from **each** vertex of n_i to **each** vertex of n_j. The following example demonstrates an example for $G = K_{1,1,2,4,4}$. The gathering of independent sets to minimize $(b - a)$ need not be distinct; for instance n_1, n_2 below could be swapped out with n_3, with no harm done.



$K_{1,1,2,4,4}$

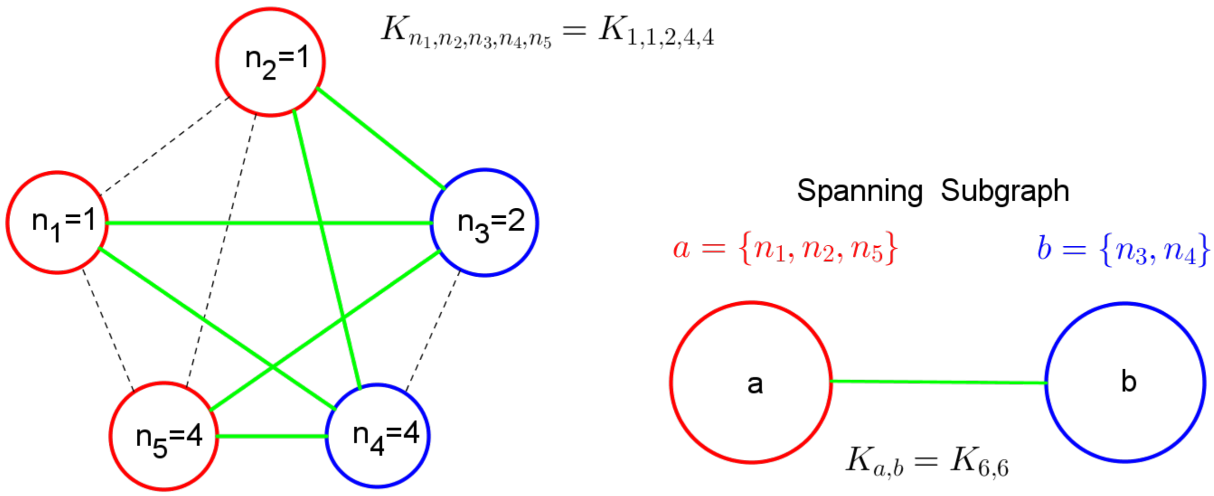
$A = \{1, 1, 4\}$ $\quad$ $B = \{2, 4\}$

$p = 2$ $\quad$ $q = 3$

$a = 6$ $\quad$ $b = 6$

$b - a = 0$ (minimized)

It is plain to see that $K_{6,6}$, the complete bipartite graph with independent sets A and B shown previously is a spanning subgraph for $G = K_{1,1,2,4,4}$, as indicated below.



In the general case, with $G = K_{n_1,n_2,\ldots,n_k}$, we have that $K_{a,b}$ is a spanning subgraph for G. Consider that G has each of the edges in $K_{a,b}$ and possibly more. Since $diam(K_{a,b}) = 2$, then for any nonadjacent vertices u and v in G, a $u - v$ path is a geodesic in G if and only if it is a geodesic in $K_{a,b}$. From this and Theorem 2.4, we have:

$$src(G) \leq src(K_{a,b}) \leq \lceil \sqrt[a]{b} \rceil$$

We return to the proof of Theorem 2.5. We have $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$ from the previous page, and are pursuing the case that $n_k \geq 2$ and $s > t$. Notice $s > t$ yields that $q \geq 2$ since $b - a > 0$. It was already shown that $src(G) \geq 2$ when $n_k \geq 2$, so from the inequality above, it is sufficient to show that $b \leq 2^a$.

Assume to the contrary, that $b > 2^a$. Order the elements of B such that $b_1 \leq b_2 \leq \ldots \leq b_q$.
<u>Claim 1:</u> $b_i \leq a$ for each i. Suppose not, to force a contradiction. Then $b_q > a$. Since $s > t$, the selections $A^* = \{b_q\}$, $B^* = A \cup B - \{b_q\}$ for A and B yield a lower nonnegative value for $b - a$, which means sets A and B were not valid, a contradiction. Thus $b_i \leq a$ for $1 \leq i \leq q$.
<u>Claim 2:</u> $b < 3a$. Again, assume this is false to force a contradiction. Then $b \geq 3a$. Since $b_1 \leq a$, we have the following: $a + b_1 \leq 2a = 3a - a \leq b - a \leq b - b_1$.
Observe that since $a + b_1 \leq b - b_1$, we could have included b_1 in set A, meaning again that choices for sets A and B were invalid, a contradiction. Then $b < 3a$.

Using these two results, we see that $2^a < b < 3a$. The only pair of integers satisfying this inequality is $(2, 5)$ In this case, since $b_i \leq 2$ we are forced to conclude that $b_1 = 1$. So b_1 could have been included in set A, meaning $(2, 5)$ is not a valid pair. So we have no integers satisfying $2^a < b < 3a$. Our working assumption that $b > 2^a$ is clearly false, leading to our desired result that $b \leq 2^a$. Therefore $src(G) \leq \lceil \sqrt[a]{b} \rceil \leq \lceil \sqrt[a]{2^a} \rceil = 2$

We conclude Theorem 2.5 by showing that if $s \leq t$, $src(G) = \lceil \sqrt[s]{t} \rceil$. Let W be the unique independent set n_k. As before, $K_{s,t}$ is a spanning subgraph of G and $src(G) \leq src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$. The claim is that $src(G) = \lceil \sqrt[s]{t} \rceil$. Once more we suppose not by means of a contradiction. So then $src(G) = \ell < \lceil \sqrt[s]{t} \rceil$, producing $t > \ell^s$. Obviously there exists a strong rainbow ℓ-coloring c of the graph G. Returning to the idea of color codes from previous work, and since $deg(w) = s$ for each of the t vertices in W, we can have at most ℓ^s dinstinct s-tuples for color codes in W. This means there are 2 vertices w_a, w_b such that $code(w_a) = code(w_b)$. Then there is no $w_a - w_b$ geodesic in G, contradicting the claim of a strong rainbow ℓ-coloring for G.
$\Rightarrow \quad src(G) = \lceil \sqrt[s]{t} \rceil \quad \square$

4.2 Rainbow Connection in Complete Multipartite Graphs

Notice we have actually shown results on strong rainbow colorings first for complete multipartite graphs. Results on rainbow colorings are next, and whereas strong rainbow connection numbers increased with the size of the graphs in Theorems 2.4 and 2.5, results regarding rainbow connection numbers in multipartite graphs are bounded. We continue, first with an upper bound on the rainbow connection number of complete bipartite graphs.

Theorem 2.6: For integers s and t with $2 \leq s \leq t$,

$$rc(K_{s,t}) = min\{\lceil \sqrt[s]{t} \rceil, 4\}$$

Proof: We first show that that for $\lceil \sqrt[s]{t} \rceil < 4$, $rc(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$. We then display a rainbow 4-coloring of $K_{s,t}$ for $\lceil \sqrt[s]{t} \rceil \geq 4$. By assumption $s \geq 2$, which means $\lceil \sqrt[s]{t} \rceil \geq 2$. So we consider cases for $\lceil \sqrt[s]{t} \rceil = 2$ and $\lceil \sqrt[s]{t} \rceil = 3$. We will let U and W be the bipartite sets of $K_{s,t}$ with $U = \{u_1, u_2, ..., u_s\}$ and $W = \{w_1, w_2, ..., w_t\}$

Case $\lceil \sqrt[s]{t} \rceil = 2$: Since $\sqrt[s]{t} \leq 2$ and by assumption $s \leq t$, we have $s \leq t \leq 2^s$. From Theorem 2.4, $src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$, so Proposition 1.1 forces

$$diam(K_{s,t}) = 2 \leq rc(K_{s,t}) \leq src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil = 2.$$

Thus $rc(K_{s,t}) = 2$.

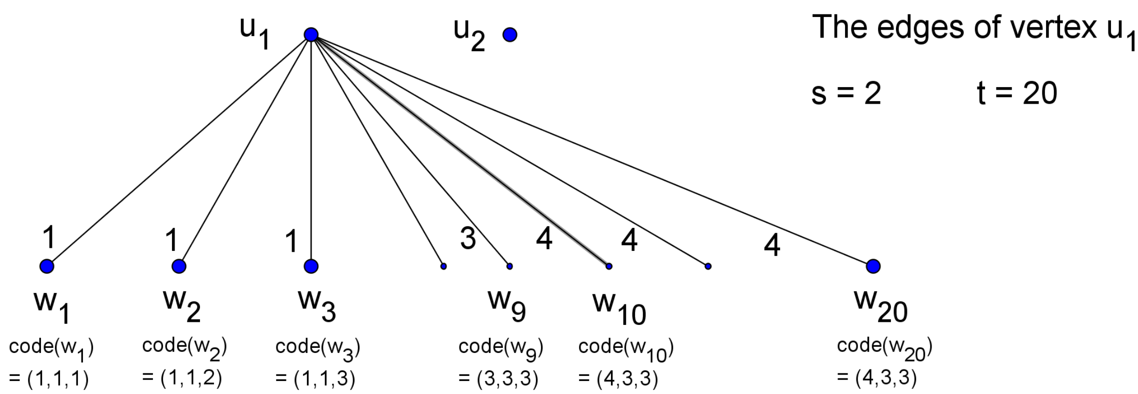
Case $\lceil \sqrt[s]{t} \rceil = 3$: Then $2^s + 1 \leq t \leq 3^s$. From Proposition 1.1 and Theorem 2.4, we have

$$diam(K_{s,t}) = 2 \leq rc(K_{s,t}) \leq src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil = 3$$

which mean $rc(K_{s,t}) \in \{2, 3\}$. Suppose by way of contradiction there is a rainbow 2-coloring of $K_{s,t}$. As in prior proofs, for $w \in W$, let $code(w)$ be the s-tuple which defines a color code for the s edges between w and U. Since $t > 2^s$, the pigeonhole principle assures us that there are two vertices $w_a, w_b \in W$ such $code(w_a) = code(w_b)$. Thus there is no rainbow $w_a - w_b$ path of length 2 in $K_{s,t}$, so that $rc(K_{s,t}) = 3$ is forced.

Case $\lceil \sqrt[s]{t} \rceil \geq 4$: In this case $t \geq 3^s + 1$. We need to prove that $rc(K_{s,t}) = 4$ Again using a contradictive proof, assume c is a rainbow 3-coloring of $K_{s,t}$, in order to show $rc(K_{s,t}) \geq 4$. We turn to the pigeonhole principle once again, to see that since $t > 3^s$, there exist w_a, w_b with $code(w_a) = code(w_b)$. Now, the only $w_a - w_b$ paths in $K_{s,t}$ are of even length. Since c is a 3-coloring, any $w_a - w_b$ path has length 2. But since $code(w_a) = code(w_b)$, we have no rainbow $w_a - w_b$ paths length 2. We conclude there are no rainbow 3-colorings, meaning $rc(K_{s,t}) \geq 4$.

Lastly, we provide a rainbow 4-coloring of $K_{s,t}$. Let $A = \{1, 2, 3\}$, $W = \{w_1, w_2, ..., w_t\}$, $W' = \{w_1, w_2, ..., w_{3^s}\}$, and $W'' = W - W'$. Match up the 3^s distinct elements of A^s to the vertices of W'. For the color codes of vertices of W'', assign a 4 to the first digit and a 3 to all remaining digits. That is, for $w \in W''$, $code(w) = (4, 3, ..., 3)$. For the coloring c we define $c(w_i u_j) = k$ if the jth coordinate of $code(w_i)$ is k. Below is an example of how the coloring of edges from just one vertex in U might appear.



Color Codes of W

Now suppose x and y are arbitrary vertices of $K_{s,t}$.

Case 1: $x, y \in W'$. Then $code(x) \neq code(y)$, which means there is some i, with $1 \leq i \leq s$ such that the ith coordinates of $code(x)$ and $code(y)$ differ, so that x, u_i, y is a rainbow path of length 2 in $K_{s,t}$.

Case 2: $x \in W'$ and $y \in W''$. The first coordinate of $code(x)$ is 1, 2, or 3 and the first coordinate of y is 4. Thus x, u_1, y is a rainbow path of length 2.

Case 3: $x, y \in W''$. There are s-tuples in the set A^s with first and second coordinates 1 and 2 respectively. Thus there exists w_i in W' such that the first coordinate of $code(w_i)$ is 1 and the first coordinate of $code(w_i)$ is 2. Then x, u_1, w_i, u_2, y is a rainbow path with colors 4, 1, 2, 3 respectively.

Case 4: $x, y \in U$. We write $x = u_i, y = u_j$ where $1 \leq i \leq j \leq s$. As before, there is some $w_i \in W'$ such that the ith and jth coordinates of w_i differ, which means x, w_i, y is a rainbow path in $K_{s,t}$.

So c is a rainbow 4-coloring of $K_{s,t}$, and $rc(G) = 4$ as claimed. $\square$

We now generalize to the rainbow connection numbers of all complete multipartite graphs.

Theorem 2.7: Let $G = K_{n_1,n_2,...,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq ... \leq n_k$. Further, let $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then:

$$rc(G) = \begin{cases} 1 & if \;\; n_k = 1 \\ 2 & if \;\; n_k \geq 2 \text{ and } s > t \\ min\{\lceil \sqrt[s]{t} \rceil, 3\} & if \;\; s \leq t \end{cases}$$

Proof: Let $n = s + t = \sum_{i=1}^{k} n_i$. If $n_k = 1$, then $n_i = 1$ for $1 \leq i \leq k-1$. We have directly from Proposition 1.1 that $rc(G) = 1$. So assume $n_k \geq 2$ and that $s > t$. By Theorem 2.5, $src(G) = 2$, which means, using Proposition 1.1 again, that $rc(G) = 2$. Finally we consider the case that $n_k \geq 2$ and $s \geq t$.

A complete graph has only independent sets of size 1, which means $G \neq K_n$. By Theorem 2.5, $src(G) = \lceil \sqrt[s]{t} \rceil$, so that $rc(G) \leq \lceil \sqrt[s]{t} \rceil$. We now show that $rc(G) \leq 3$ by providing a rainbow 3-coloring of G. Label the k partite sets of G by $V_1, V_2, ..., V_k$ where

$$V_i = \{v_{i,1}, v_{i,2}, ..., v_{i,n_i}\} \qquad \text{for } 1 \leq i \leq k.$$

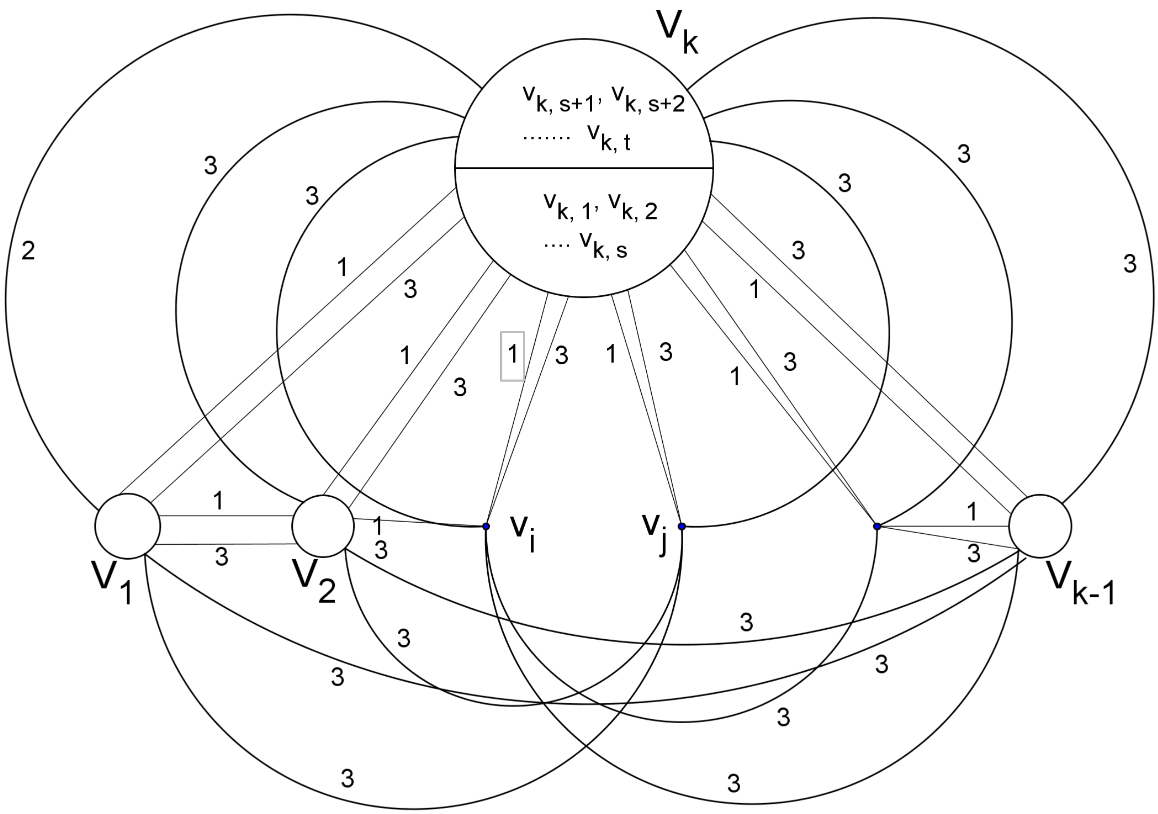
Further, let

$$U = V_1 \cup V_2 \cup ... \cup V_{k-1} = \{u_1, u_2, ..., u_s\}.$$

such that $v_{i,j} = u_{t_i+j}$ where $t_i = \sum_{j=1}^{i-1} n_j$ for $1 \leq i \leq k-1$. Observe that $|U| = s$. Now we define a coloring c on G by:

$$c(e) = \begin{cases} 1 & if \;\; e = v_{i,j}v_{i+1,j} \text{ for } 1 \leq i \leq k-2 \text{ and } 1 \leq j \leq n_i, \text{ or} \\ & if \;\; e = u_\ell v_{k,\ell} \text{ for } 1 \leq \ell \leq s, \\ 2 & if \;\; e = v_{1,j}v_{k,\ell} \text{ for } 1 \leq j \leq n_1 \text{ and } s+1 \leq \ell \leq t, \\ 3 & \text{otherwise} \end{cases}$$

We will first present a visual aid, followed by the proof that $c : E(G) \to \{1, 2, 3\}$ is indeed a rainbow 3-coloring of G. We continue with the illustrative technique used previously. We construct the complete graph K_k where each vertex represents a partite set n_i from G. Recall that an edge between nodes n_i, n_j in the new complete graph reflects an edge between each vertex of n_i with each vertex of n_j.



Let $x, y \in V(G)$. If x and y are adjacent we are done, so suppose x and y are 2 nonadjacent vertices of G. Then $x, y \in V_i$ for some i with $1 \leq i \leq k$. Label $x = v_{i,p}$ and $y = v_{i,q}$ where, without loss, $1 \leq p < q \leq n_i$. Take first the case where $1 \leq i \leq k-1$. Since $p \neq q$, $c(xv_{i+1}) \neq c(y, v_{i+1})$, which means x, v_{i+1}, y is a rainbow path. If instead $i = k$, we have 3 cases.

Case $1 \leq p < q \leq s$: On observation, $\{x, u_p, y\}$ is a rainbow path colored 1 and 3.

Case $s+1 \leq p < q \leq t$: Then $x, v_{1,1}, v_{2,1}, y$ is a rainbow path colored 2, 1 and 3.

Case $1 \leq p \leq s < q \leq t$: Here we have that $x, v_{1,1}, y$ is a rainbow path with edges colored 3 and 2 respectively.

In all cases, there exists a rainbow $x - y$ path. So $rc(G) \leq 3$. To show $rc(G) \geq 3$, suppose to the contrary that $rc(g) = 2$ and let c' be a strong rainbow 2-coloring, guaranteed by Proposition 1.1). Consider the color codes which can be assigned to each vertex $w \in W$, where $a_i = c(u_i w) \in \{1, 2\}$ for $1 \leq i \leq s$. We once again invoke the pigeonhole principle, using the fact that $t > 2^s$, to see there exist $w_a, w_b \in W$ such that $code(w_a) = code(w_b)$. The colors, then, of any length 2 $w_a - w_b$ path in G are identical. So there is no rainbow $w_a - w_b$ geodesic. By this contradiction we conclude that $rc(G) \geq 3$, forcing that $rc(G) = 3$. This concludes the proof of Theorem 2.7. $\square$

4.3 Additional Results

We summarize a few results. Having seen that $rc(G) \leq src(G)$ for each nontrivial connected graph G, then by Proposition 1.1, we have that for every positive integer a and every tree T with a edges, $rc(T) = src(T) = a$. Further, if $a \in \{1, 2\}$, $rc(G) = a$ if and only if $src(G) = a$. If $a = 3$ and $b \geq 4$, then by Propositions 2.2 and 2.3, $rc(W_{3b}) = 3$ and $src(W_{3b}) = b$.

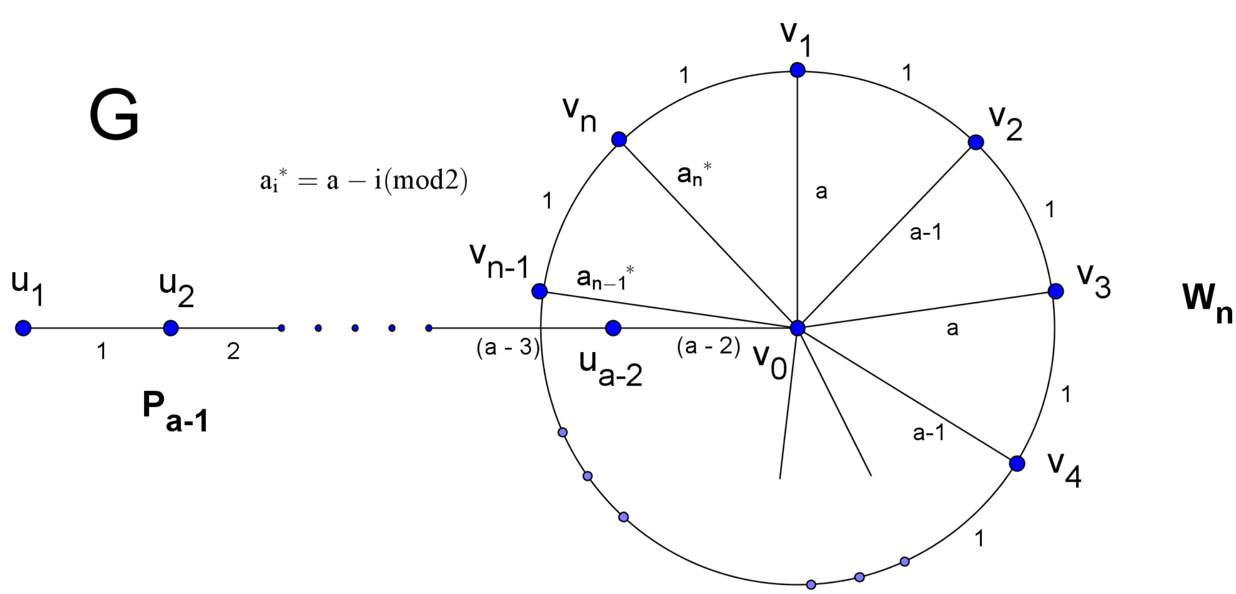
So then, we next assume that $a \geq 4$ and gain the following result:

Theorem 3.1: Let a and b be integers with $a \geq 4$ and $b \geq (5a-6)/3$. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$.

Proof: Let $n = 3b - 3a + 6$ and W_n be the wheel constructed from C_n with a new vertex v_0, adjacent to each vertex in C_n. As usual, $C_n = v_1, v_2, ..., v_n, v_1$ where $v_i \neq v_j$ for $2 \leq i \leq n$. Now let G be the graph constructed from W_n and the path $P_{n-1} = u_1, u_2, ..., u_{a-1}$ such that $u_{a-1} = v_0$.

We will first show that $rc(G) = a$. For $a \geq 4$, we have $b \geq (5a-6)/3 > (5a-2a)/3 = a$. Then we may conclude that $b - a \geq 1$, yielding $n = 3b - 3a + 6 \geq 1 + 6 = 7$. Thus $rc(W_n) = 3$, by Proposition 2.2. Next, define a coloring $c : E(G) \to \{1, 2, ...a\}$ by:

$$c(e) = \begin{cases} i & if \;\; e = u_iu_{i+1} \text{ for } 1 \leq i \leq a-2 \\ a & if \;\; e = v_iv_0 \text{ and } i \text{ is odd }, \\ a-1 & if \;\; e = v_iv_0 \text{ and } i \text{ is even }, \\ 1 & \text{otherwise} \end{cases}$$



To show c is a rainbow a-coloring, let $x, y \in G$ and consider by cases.

Case $x, y \in P_{a-1}$: Since each edge in P_{a-1} has a distinct color, there is a rainbow $x-y$ path.

Case $x, y \in W_n$: Write $x = v_i$ and $y = v_j$. Without loss, say $i < j$. If $c(v_iv_0) \neq c(v_jv_0)$ then v_i, v_0, v_j is a rainbow $x-y$ path of length 2. Otherwise v_i, v_{i+1}, v_0, v_j is a rainbow $x-y$ path of length 3. (The edges of W_n alternate and we ruled out $i = n$, $j = 1$)

Case $x \in P_{a-1}, y \in W_n$: Write $x = u_i$ and $y = v_j$. Observe that $u_i, u_{i+1}, ..., v_0$ is a rainbow $x - v_0$ path. Since none of the spokes of the wheel W_n share a color with P_{a-1}, we have that $P = u_i, u_{i+1}, ..., v_0, v_j$ is a rainbow $x-y$ path in G. Thus c is a rainbow a-coloring of G, meaning $rc(G) \leq a$.

To show $rc(G) = a$ we show that there is no $(a-1)$ rainbow coloring of G. Assume to the contrary, that c^* is a rainbow $(a-1)$ coloring of G. We can assume without loss that $c^*(u_iu_{i+1}) = i$ for $1 \leq i \leq a-2$, since each edge in P_{a-1} must have distinct colors. Clearly the path $u_1, u_2, ..., v_0$ uses $(a-1)$ colors. Since any $u_1 - v_i$ path is distinct for $1 \leq i \leq n$, there is no choice but that $c(v_iv_0) = a-1$ for $1 \leq i \leq n$.

Observe that $n = 3b - 3a + 6 \geq 3\left(\frac{5a-6}{3}\right) - 3a + 6 = 5a - 6 - 3a + 6 = 2a$. Since $n \geq 2a$, any $v_1 - v_{a+1}$ path has length a or greater. Thus any $v_1 - v_{a+1}$ path has 2 edges which share a color, meaning there is no rainbow $v_1 - v_{a+1}$ path in G. Apparently there does not exist an $(a-1)$ coloring of G, which concludes the proof that $rc(G) = a$.

We again refer to the illustration of graph G on the previous page as we move to the proof that $src(G) = b$. First consider that $\lceil \frac{n}{3} \rceil = \frac{n}{3} = a - b + 2$. By Theorem 2.3, then, with W_n as a subgraph of G and $n \geq 7$, $src(W_n) = a - b + 2$. Let c_1 be a strong rainbow $(b-a+2)$ coloring of W_n. Define the coloring c on G by

$$c(e) = \begin{cases} c_1(e) & if \;\; e \in E(W_n) \\ b - a + 2 + i & if \;\; e = u_iu_{i+1} \text{ for } 1 \leq i \leq a-2 \end{cases}$$

By assumption, c_1 is already a strong rainbow $(b-a+2)$ coloring on W_n. Moreover, since the edges of P_{a-1} and W_n share no colors, all $u_i - v_j$ geodesics are rainbow geodesics. Lastly, noting that there are $(a-2)$ colors used on the edges of P_{a-1} and $(b-a+2)$ colors used on the edges of W_n, we see that c is in fact a strong rainbow b-coloring of G.

Having satisfied $src(G) \leq b$ we show that $src(G) \geq b$ by again supposing for the purpose of a contradiction that there is a strong rainbow $(b-1)$ coloring on G. Let c^* be such a coloring, and assume without loss that $c^*(u_iu_{i+1}) = i$ for $1 \leq i \leq a-2$. Further, let $C = \{a-1, a, ..., b-1\}$ and $S = \{v_0v_j : 1 \leq j \leq 3b - 3a + 6\}$. Note that $|C| = b - a + 1$ and $|S| = 3b - 3a + 6$. Returning to the idea that $u_i - v_j$ geodesics are distinct, it must be that $c^*(v_0v_j) \in C$. It was shown previously that no more than 3 edges of the wheel can share a color without losing a strong rainbow coloring. Hence the $(b-a+1)$ colors of C can color at most $3(b-a+1) = 3b - 3a + 3$ edges. Since we have $n = 3b - 3a + 6$ edges to color in $|S|$, we cannot have a strong rainbow $(b-1)$ coloring. Therefore, $src(G) = b$ as desired. $\square$

References:

[1] West, Douglas B. “Introduction to Graph Theory”. Pearson Education, Inc., Second Edition, 2001.

[2] Godsil, Roye. “Algebraic Graph Theory”. Springer, 2001.

[3] Hujdurovic, Kutnar, Marusic. “Cubic Cayley Graphs and Snarks”. 2012 http://www.math.vanderbilt.edu ∼ shanks2012/talk/Kutnar_Shanks2012.pdf

[4] Chartrand, Johns, McKeon, Zhang “Rainbow Connection Number in Graphs”, Mathematica Bohemica, p 85-98, 133 (2008)